From Model-Based to Data-Driven Control of Network Dynamics

Giacomo Baggio

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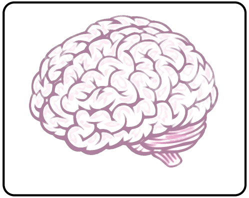
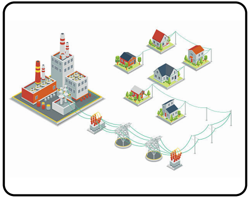
October 14, 2020

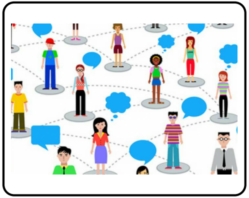
It's a dynamical, networked, and complex world !



biology



engineering



social science

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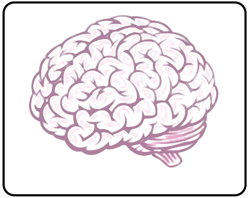
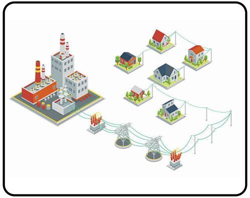
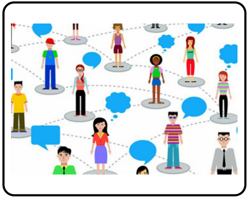


$x(t)$ = neural activity



$x(t)$ = power consumption



$x(t)$ = individual opinions

1. dynamical processes: $\dot{x}(t) = f(x(t))$ or $x(t+1) = f(x(t))$

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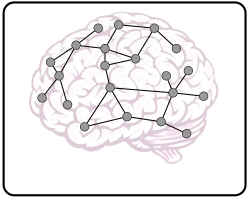

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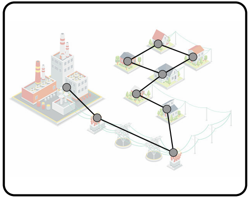

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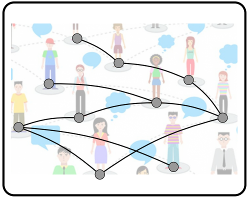

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1. dynamical processes: $\dot{x}(t) = f(x(t))$ or $x(t+1) = f(x(t))$
2. many "simple" units interconnected through a network: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

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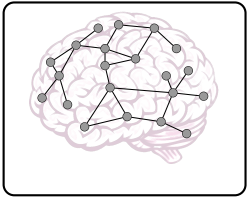
cognition, seizures



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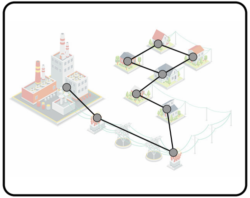
desync, blackouts



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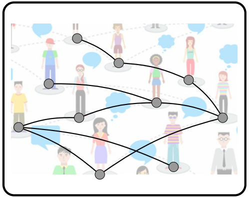
consensus, polarizations



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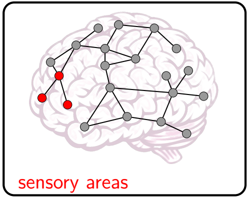

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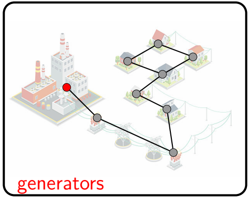

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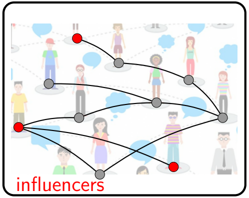

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3. emergence of complex collective phenomena/behaviors
4. presence of nodes with control authority

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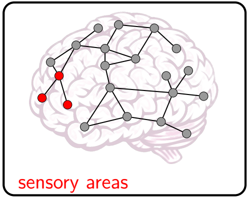

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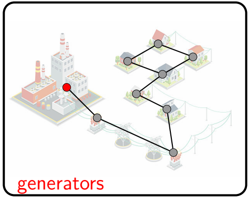

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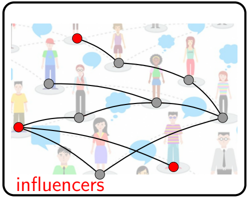

$x(t)$ = individual opinions

Analyze, predict, and *control* dynamics over large-scale networks

Disclaimer

dynamics = linear dynamics

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(t+1) = Ax(t) + Bu(t)$$

$$x(0) = x_0 \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n \times n}, \, B \in \mathbb{R}^{n \times m}$$

Roadmap

Network controllability:

- the structural approach
- the "practical" approach

Non-normal dynamics and network structure

Data-driven control of networks

Conclusions & open challenges

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Conclusions & open challenges

Control theory meets network science

12 May, 2011

ARTICLE

doi:10.1038/nature10011

Controllability of complex networks

Yang-Yu Liu[1,2], Jean-Jacques Slotine[3,4] & Albert-László Barabási[1,2,5]

Control theory meets network science

12 May, 2011

ARTICLE

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Controllability of complex networks

ARTICLE

Received 7 Apr 2015 | Accepted 19 Aug 2015 | Published 1 Oct 2015 DOI: 10.1038/ncomms9414 OPEN

Controllability of structural brain networks

Shi Gu[1,2], Fabio Pasqualetti[3], Matthew Cieslak[4], Qawi K. Telesford[2,5], Alfred B. Yu[5], Ari E. Kahn[2], John D. Medaglia[2], Jean M. Vettel[4,5], Michael B. Miller[4], Scott T. Grafton[4] & Danielle S. Bassett[2,6]

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Controllability of structural brain networks

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Nodal Dynamics, Not Degree Distributions, Determine the Structural Controllability of Complex Networks

Noah J. Cowan[1]*, Erick J. Chastain[2], Daril A. Vilhena[3], James S. Freudenberg[4], Carl T. Bergstrom[3,5]

Control theory meets network science

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Nodal Dynamics, Not Degree Distributions, Determine the Structural Controllability of Complex Networks

T. Bergstrom[3,5]

BRIEF COMMUNICATIONS ARISING

Few inputs can reprogram biological networks

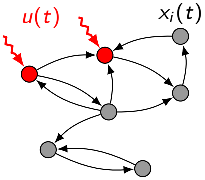
ARISING FROM Y. Liu, J. Slotine & A. Barabási *Nature* **473,** 167–173 (2011)

...

Network controllability: setting

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

$A =$ network matrix



dynamical system with state $x(t)$

$+$ control $u(t)$

$$x(t+1) = Ax(t) + Bu(t)$$

B (typically) selects a subset of $\mathcal{V}$

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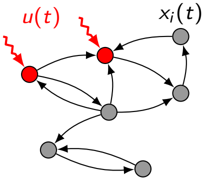

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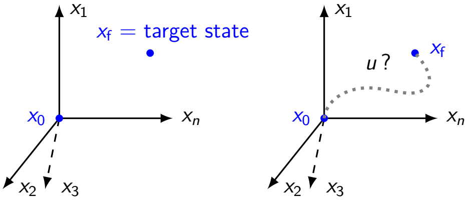

controllability =

$\exists\, u(t), T\colon\ x(0) = x_0,\ x(T) = x_f,\ \forall x_0, x_f$

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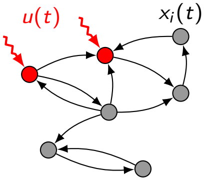

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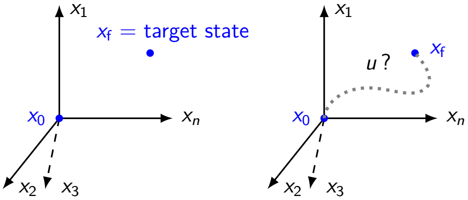

controllability $=$

$\exists\, u(t), T\colon\; x(0) = x_0,\; x(T) = x_{\mathrm{f}},\; \forall x_0, x_{\mathrm{f}}$

iff

$$\operatorname{rank} \underbrace{\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}}_{\mathcal{C}} = n$$

Kalman rank condition (1963)

Network controllability: the structural approach

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

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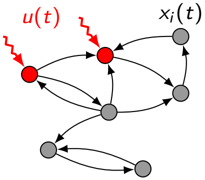

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structural controllability $=$

$\exists$ weights such that network is controllable

$\Downarrow$

controllability for *almost all* choices of weights!

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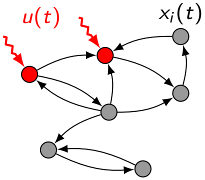

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controllability for *almost all* choices of weights!

- ✓ captures the role of network topology
- ✓ can be checked via graphical conditions

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. AC-19, NO. 3, JUNE 1974 201

Structural Controllability

CHING-TAI LIN, MEMBER, IEEE

Structural network controllability: some relevant questions (w/ answers)

- How does the network structure affect structural controllability? [Liu et al., 2011]
  - ▷ dense, homogeneous networks are "easier" to control (require fewer inputs)
  - ▷ choice of hubs as control nodes is not "optimal"

- What is the minimum set of control nodes that guarantees controllability?
  - ▷ polynomial time algorithms [Pequito et al., 2016]

  ("standard" controllability: NP-hard problem [Olshevsky, 2014] !)

...however edge weights do matter !

controllable networks might be uncontrollable in practice !!

Network controllability: the "practical" approach

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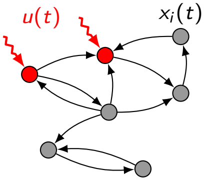

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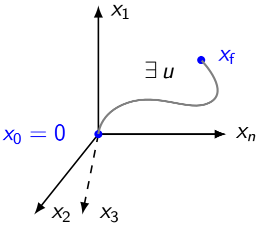

How much energy is needed?

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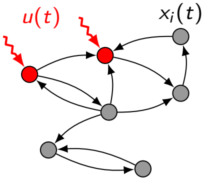

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T-steps controllability Gramian:

$$\mathcal{W}_T = \sum_{k=0}^{T-1} A^k B B^\top (A^\top)^k$$

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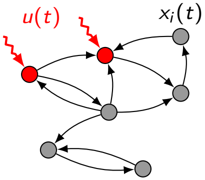

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$$u^\star(t) = B^\top (A^\top)^{T-t-1} \mathcal{W}_T^{-1} x_{\mathrm{f}}$$

$$t = 0, 1, \ldots, T-1$$

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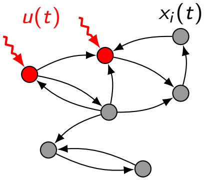

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energy needed to reach x_f in T steps:

$$\sum_{t=0}^{T-1} \|u^\star(t)\|^2 = x_\mathrm{f}^\top \mathcal{W}_T^{-1} x_\mathrm{f}$$

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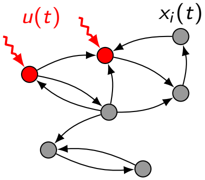

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scalar metrics:

$\lambda_{\min}^{-1}(\mathcal{W}_T) =$ worst-case control energy

$\operatorname{tr}(\mathcal{W}_T^{-1}) =$ average control energy

$1/\det(\mathcal{W}_T) =$ "volumetric" control energy

$\vdots$

Network controllability: the "practical" approach

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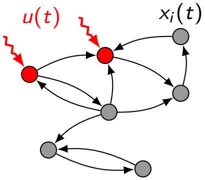

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$\lambda_{\min}(\mathcal{W}_T) \downarrow\downarrow \Longrightarrow$ control energy $\uparrow\uparrow$

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Difficult-to-control networks

[Pasqualetti et al., 2014], [Bof et al., 2016], [Olshevsky, 2016],...

Theorem: Let A be diagonalizable with eigenvector matrix V, and (Schur) stable. Then, for all $T \in \mathbb{N}_{>0}$:

$$\lambda_{\min}(\mathcal{W}_T) \leq \min \left\{ \|V\|^2 \|V^{-1}\|^2 \frac{\rho(A)^{2(\frac{n}{m}-1)}}{1-\rho(A)^2}, \frac{4\mu(A)^{2(\frac{n}{m}-1)}}{1-\mu(A)^2} \right\}$$

where $\rho(A) = \max_{\lambda \in \sigma(A)} \lambda$ (spec. radius) and $\mu(A) = \max_{\|x\|=1} |x^\top A x|$ (num. radius).

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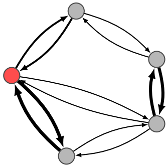

If A is stable, *normal*, and m is fixed and independent of n:

$$\lambda_{\min}(\mathcal{W}_T) \leq K^n, \text{ with } 0 < K < 1$$

control energy grows exponentially fast with n !!

Easy-to-control networks?

If A is stable and m is fixed and independent of n, are there networks such that

$$\lambda_{\min}(\mathcal{W}_T) \geq L,\ L > 0, \text{ for all } n \text{ ??}$$

N.B. Such networks must satisfy $\|V\|\|V^{-1}\| \gg 1$ and/or $\mu(A) > 1$

$\Longrightarrow$ *strong non-normality* !

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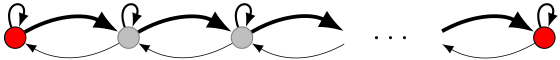
$\Longrightarrow$ *strong non-normality* !

Short answer: *Yes*

but a characterization of these networks is still largely an open problem !

An easy-to-control network

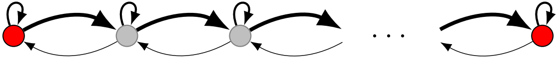
[Pasqualetti and Zampieri, 2015]



$$A = \begin{bmatrix} a & b & 0 & \cdots & 0 \\ c & a & b & \cdots & 0 \\ 0 & c & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a & b \\ 0 & \cdots & \cdots & c & a \end{bmatrix}$$

An easy-to-control network

[Pasqualetti and Zampieri, 2015]



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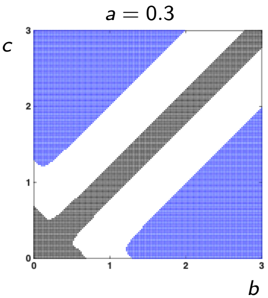

Theorem: A Toeplitz line network is easy to control if one of the following holds:

- $\frac{a(b+c)}{4bc} \leq 1$ and $1 < (b-c)^2(1-\frac{a^2}{4bc})$
- $\frac{a(b+c)}{4bc} > 1$ and $1 \leq c+b-a$

Loosely speaking...

$$\boxed{\mathcal{W}_T \uparrow\uparrow \iff \mathcal{W}_T^{-1} \downarrow\downarrow \iff \text{control energy} \downarrow\downarrow}$$

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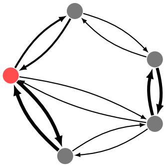
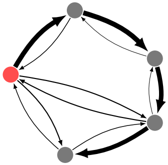


A normal $\implies \mathcal{W}_T$ "small"
$\implies$ difficult to control



A non-normal $\implies \mathcal{W}_T$ (potentially) "large"
$\implies$ (potentially) easy to control

Roadmap

Network controllability:
- the structural approach
- the “practical” approach

Non-normal dynamics and network structure

Data-driven control of networks

Conclusions & open challenges

Matrix non-normality

$A \in \mathbb{R}^{n \times n}$

normal matrices

$AA^\top = A^\top A$

non-normal matrices

$AA^\top \neq A^\top A$

Matrix non-normality

$A \in \mathbb{R}^{n \times n}$

normal matrices	non-normal matrices
$AA^\top = A^\top A$	$AA^\top \neq A^\top A$
symmetric, skew-symmetric, orthogonal, circulant...	*...all the rest!*

Matrix non-normality

$A \in \mathbb{R}^{n \times n}$

normal matrices

$AA^\top = A^\top A$

$$A = U^* D U,$$

U unitary, D diagonal

non-normal matrices

$AA^\top \neq A^\top A$

(for A diagonalizable)

$$A = V^{-1} D V,$$

V *not* unitary, D diagonal

Matrix non-normality

$$A \in \mathbb{R}^{n\times n}$$

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$$AA^\top = A^\top A$$

fully described by spectrum

$$\sigma(A) = \{\lambda_i\}_{i=1}^n$$

non-normal matrices

$$AA^\top \neq A^\top A$$

described by ε-pseudospectrum

($\approx$ perturbed spectrum)

$$\sigma_\varepsilon(A) = \{\lambda \in \sigma(A+E), \quad E \in \mathbb{C}^{n\times n}, \|E\| \leq \varepsilon\}$$

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small perturbations of the entries

$\Downarrow$

small perturbations of the spectrum

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small perturbations of the entries

$\Downarrow$

(possibly) *large* perturbations of the spectrum

Matrix non-normality

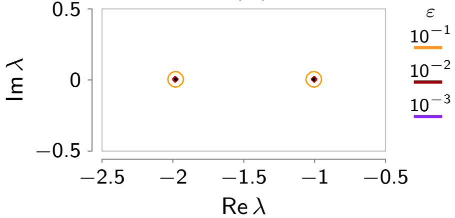
$A \in \mathbb{R}^{n \times n}$

normal matrices

$$AA^\top = A^\top A$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$\sigma_\varepsilon(A)$

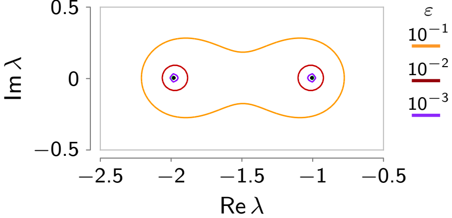


non-normal matrices

$$AA^\top \neq A^\top A$$

$$A = \begin{bmatrix} -1 & 0 \\ 5 & -2 \end{bmatrix}$$

$\sigma_\varepsilon(A)$



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$\dot{x}(t) = Ax(t),\ x(0) = x_0$

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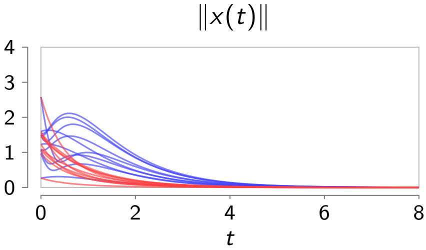

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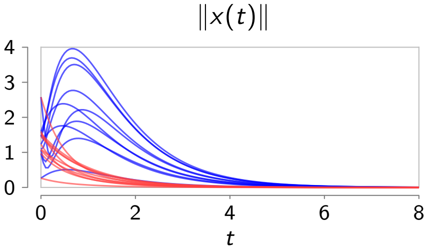


$$A = \begin{bmatrix} -1 & 0 \\ 10 & -2 \end{bmatrix}$$

Matrix non-normality

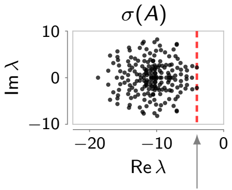
$A \in \mathbb{R}^{n \times n}$

normal matrices

$AA^\top = A^\top A$

non-normal matrices

$AA^\top \neq A^\top A$



$\alpha(A) := \max_{\lambda \in \sigma(A)} \operatorname{Re} \lambda \to$ *asymptotic* behavior

"non-normality" of $A \to$ *transient* behavior

Non-normal network dynamics

Non-normality has been shown to play a key role in many real networks

Non-normal network dynamics

Non-normality has been shown to play a key role in many real networks

SCIENCE ADVANCES | RESEARCH ARTICLE

NETWORK SCIENCE

Structure and dynamical behavior of non-normal networks

Malbor Asllani[1,2], Renaud Lambiotte[1], Timoteo Carletti[2]*

frontiers in ECOLOGY AND EVOLUTION

ORIGINAL RESEARCH ARTICLE
published: 04 June 2014
doi: 10.3389/fevo.2014.00021

Reactivity and stability of large ecosystems

Si Tang[1] and Stefano Allesina[1,2]*

[1] Department of Ecology and Evolution, University of Chicago, Chicago, IL, USA
Computation Institute, University of Chicago, Chicago, IL, USA

PHYSICAL REVIEW RESEARCH **2**, 023333 (2020)

Universal transient behavior in large dynamical systems on networks

Wojciech Tarnowski,[1] Izaak Neri,[2] and Pierpaolo Vivo[2]
[1]*Institute of Theoretical Physics, Jagiellonian University, S. Lojasiewicza 11, PL 30-348 Kraków, Poland*
[2]*Department of Mathematics, King's College London, Strand, London, WC2R 2LS, United Kingdom*

(Received 2 September 2019; revised manuscript received 24 March 2020; accepted 11 May 2020; published 12 June 2020)

SCIENCE ADVANCES | RESEARCH ARTICLE

NETWORK SCIENCE

Efficient communication over complex dynamical networks: The role of matrix non-normality

Giacomo Baggio[1], Virginia Rutten[2,3], Guillaume Hennequin[4]*, Sandro Zampieri[1]*†

PHYSICAL REVIEW E **86**, 011909 (2012)

Non-normal amplification in random balanced neuronal networks

Guillaume Hennequin,* Tim P. Vogels, and Wulfram Gerstner
School of Computer and Communication Sciences and Brain-Mind Institute, Ecole Polytechnique Fédé[rale] de Lausanne, 1015 EPFL, Switzerland

(Received 13 April 2012; published 11 July 2012)

Non-normality and non-monotonic dynamics in complex reaction networks

Zachary G. Nicolaou,[1] Takashi Nishikawa,[1,2] Schuyler B. Nicholson,[3,4] Jason R. Green,[3,4,5] and Adilson E. Motter[1,2]
[1]*Department of Physics and Astronomy, Northwestern University, Evanston, IL 60208, USA*
[2]*Northwestern Institute on Complex Systems, Northwestern University, Evanston, IL 60208, USA*
[3]*Department of Chemistry, University of Massachusetts Boston, Boston, MA 02125, USA*
[4]*Center for Quantum and Nonequilibrium Systems, University of Massachusetts Boston, Boston, MA 02125, USA*
[5]*Department of Physics, University of Massachusetts Boston, Boston, Massachusetts 02125, USA*

How to measure non-normality?

$\mathcal{N}$ = set of normal matrices

$$\min_{N \in \mathcal{N}} \|A - N\|$$

$$\|AA^\top - A^\top A\|$$

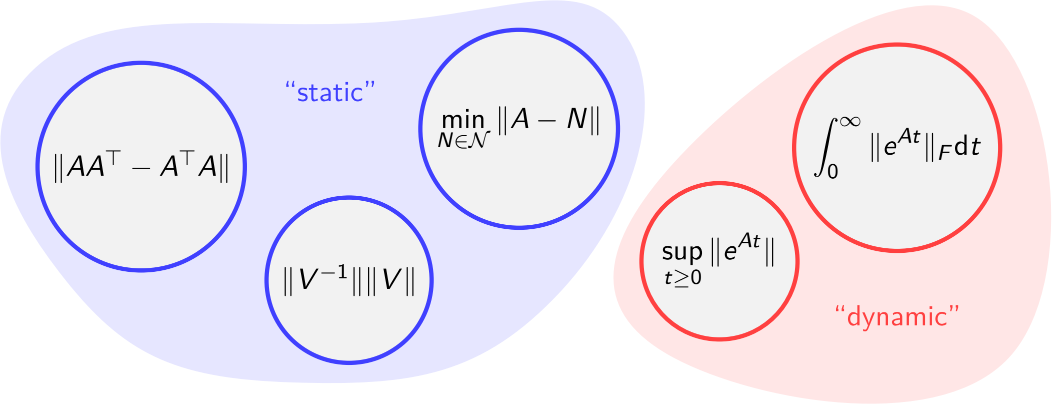
$$\int_0^\infty \|e^{At}\|_F \mathrm{d}t$$

$$\sup_{t \geq 0} \|e^{At}\|$$

$$\|V^{-1}\|\|V\|$$

V = eigenvector matrix of A

How to measure non-normality?



How to measure non-normality?

"static"

$\|AA^\top - A^\top A\|$

$\min_{N \in \mathcal{N}} \|A - N\|$

$\|V^{-1}\|\|V\|$

"dynamic"

$\int_0^\infty \|e^{At}\|_F \mathrm{d}t$

$\sup_{t \geq 0} \|e^{At}\|$

...and many more

[Trefethen and Embree, Princeton (2005)]

A link to network structure for positive networks

[Baggio and Zampieri, 2018]

$$\dot{x}(t) = Ax(t) + Bu(t)$$

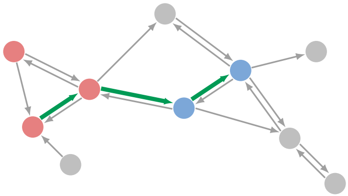
$$y(t) = Cx(t)$$

A stable and Metzler

(Re$[\sigma(A)] < 0$ and $A_{ij} > 0$, $i \neq j$)

$d(\mathcal{K}, \mathcal{T})$ = relative diameter

shortest path length between two most distant nodes $v_i \in \mathcal{K}$ and $v_j \in \mathcal{T}$



$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

$\mathcal{K} = \{$ ● $\}$

$\mathcal{T} = \{$ ● $\}$

A link to network structure for positive networks

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A stable and Metzler

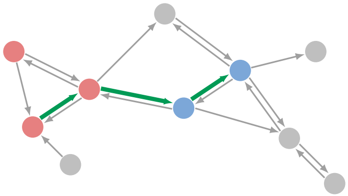
$(\mathrm{Re}[\sigma(A)] < 0$ and $A_{ij} > 0,\ i \neq j)$

non-normality $\uparrow\uparrow$ $\Longleftrightarrow$ $d(\mathcal{K}, \mathcal{T}) \uparrow\uparrow$

$(\sup_{t \geq 0} \|Ce^{At}B\|)$ + directionality

$d(\mathcal{K}, \mathcal{T})$ = relative diameter

shortest path length between two most distant nodes $v_i \in \mathcal{K}$ and $v_j \in \mathcal{T}$



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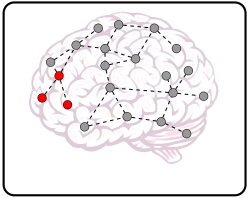
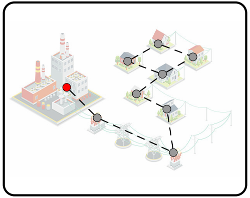
Controlling networks from data

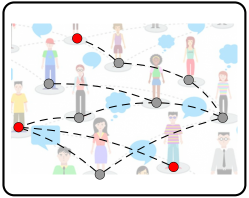
Network structure may be uncertain and/or changing over time !



$x(t)$ = neural activity



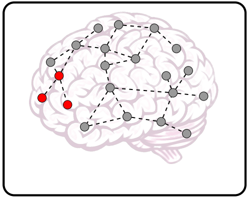
$x(t)$ = power consumption



$x(t)$ = individual opinions

Controlling networks from data

Network structure may be uncertain and/or changing over time !



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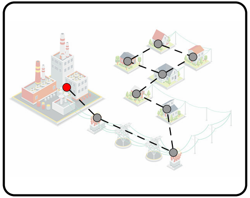

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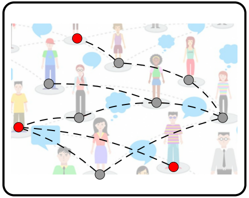

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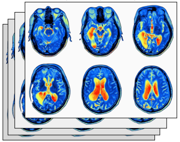

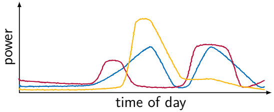

However, there's plenty of data out there...

Can we control a network *directly from data*?

The data-driven minimum-energy control problem

$$x(t+1) = \boxed{\color{red}{?}}\, x(t) + \boxed{\color{red}{?}}\, u(t), \;\; x(0) = 0$$

$x_{\mathrm{f}} \in \mathbb{R}^n$ controllable in T steps from $x(0) = 0$

The data-driven minimum-energy control problem

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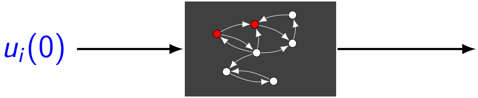
$x_f \in \mathbb{R}^n$ controllable in T steps from $x(0) = 0$

i-th control experiment



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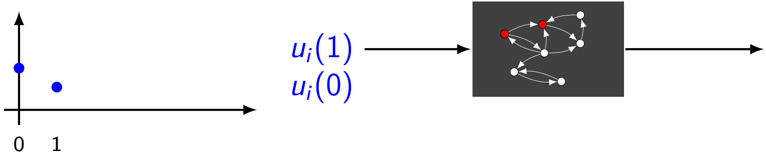

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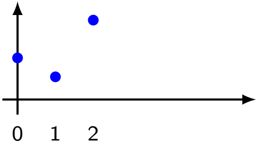
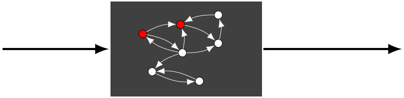


$u_i(2)$
$u_i(1)$
$u_i(0)$



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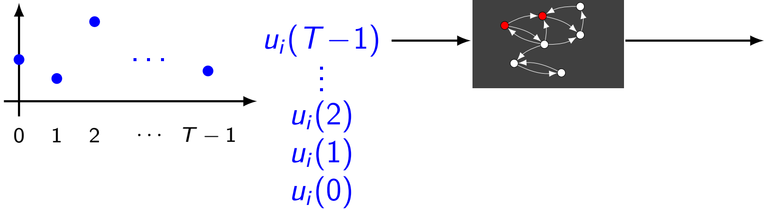

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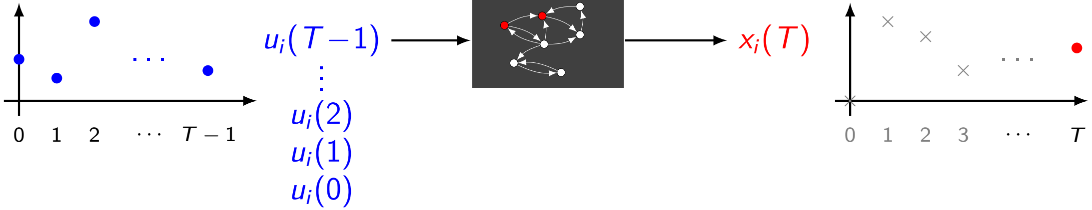

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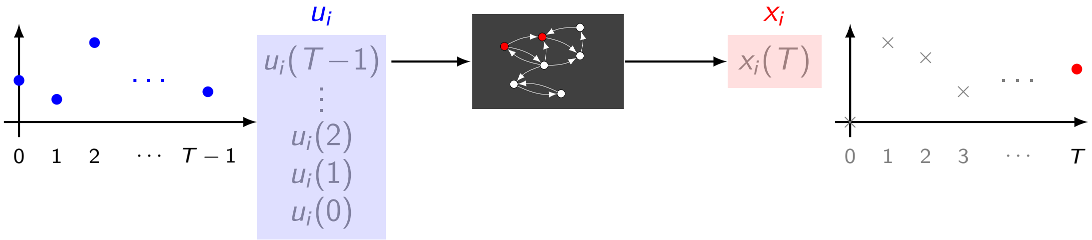
Experimental data:

$$U = [u_1 \;\; u_2 \;\; \cdots \;\; u_N]$$

$$X = [x_1 \;\; x_2 \;\; \cdots \;\; x_N]$$

i-th control experiment



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Task: compute minimum-energy control $u^\star(t)$ to reach x_f in T steps *from data*

The data-driven minimum-energy control problem

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non-optimal

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non-optimal

Task: compute minimum-energy control $u^\star(t)$ to reach x_f in T steps *from data*

directly = without identifying the system!

Data-driven minimum-energy control inputs

1

$$\begin{aligned} \alpha^\star = \arg\min_{\alpha \in \mathbb{R}^N} & \; \|U\alpha\|^2 \\ \text{s.t.} & \; x_f = X\alpha \end{aligned}$$

Data-driven minimum-energy control inputs

(1)

$$\alpha^\star = \arg\min_{\alpha\in\mathbb{R}^N} \|U\alpha\|^2 \quad \text{s.t. } x_\mathrm{f} = X\alpha$$

if U full row rank →

$$u^\star = \begin{bmatrix} u^\star(T-1) \\ \vdots \\ u^\star(0) \end{bmatrix} = U\alpha^\star$$

$$= (I - UK_X(UK_X)^\dagger)UX^\dagger x_\mathrm{f}$$

$$K_X = \text{basis of } \ker(X)$$

Data-driven minimum-energy control inputs

1

$$\alpha^\star = \arg\min_{\alpha \in \mathbb{R}^N} \|U\alpha\|^2 \quad \text{s.t. } x_f = X\alpha$$

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$$= (I - UK_X(UK_X)^\dagger)UX^\dagger x_f$$

$K_X =$ basis of $\ker(X)$

2

$$C^\star = \arg\min_{C \in \mathbb{R}^{n \times mT}} \|X - CU\|_F$$

if U full row rank $\longrightarrow$

$$u^\star = (C^\star)^\dagger x_f = (XU^\dagger)^\dagger x_f$$

Data-driven minimum-energy control inputs

①

$$\alpha^\star = \arg\min_{\alpha \in \mathbb{R}^N} \|U\alpha\|^2 \quad \text{s.t. } x_f = X\alpha$$

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$$= (I - UK_X(UK_X)^\dagger)UX^\dagger x_f$$

$$K_X = \text{basis of } \ker(X)$$

---

②

$$C^\star = \arg\min_{C \in \mathbb{R}^{n \times mT}} \|X - CU\|_F$$

if U full row rank $\longrightarrow$

$$u^\star = (C^\star)^\dagger x_f = (XU^\dagger)^\dagger x_f$$

$N = mT$ linearly independent experiments suffice to reconstruct $u^\star$

Approximate data-driven minimum-energy control inputs

$$③\ M^\star = \arg\min_{M \in \mathbb{R}^{mT \times n}} \|MX - U\|_F \longrightarrow \widehat{u} = M x_\mathrm{f} = UX^\dagger x_\mathrm{f}$$

Approximate data-driven minimum-energy control inputs

$$\textcircled{3}\ M^\star = \arg \min_{M \in \mathbb{R}^{mT \times n}} \|MX - U\|_F \longrightarrow \hat{u} = Mx_f = UX^\dagger x_f$$

$\hat{u}$ sub-optimal solution ($\hat{u} \neq u^\star$)

Approximate data-driven minimum-energy control inputs

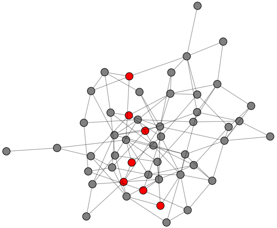
$$\textcircled{3}\; M^\star = \arg \min_{M \in \mathbb{R}^{mT \times n}} \|MX - U\|_F \longrightarrow \quad \widehat{u} = M x_{\mathrm{f}} = UX^\dagger x_{\mathrm{f}}$$

$\widehat{u}$ sub-optimal solution ($\widehat{u} \neq u^\star$), however...

Theorem: If U has i.i.d. entries with zero-mean and finite variance, then as the number of data grows ($N \to \infty$)

$$\widehat{u} \xrightarrow{\text{a.s.}} u^\star.$$

A numerical example

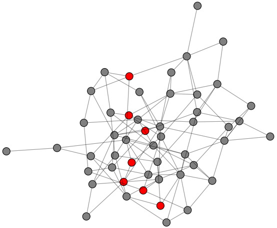


A = adjacency matrix of
Erdös-Rényi graph $p_{\text{edge}} = 0.1$

$n = 50$ nodes, $T = 10$,
$m = 7$ (rand. chosen) control nodes

U_{ij} i.i.d. r.v.'s, $\mathbb{E}[U_{ij}] = 0$, x_{f} rand. chosen

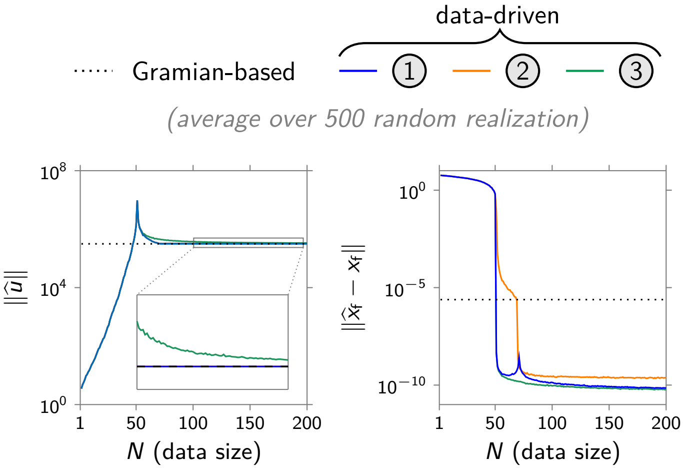
A numerical example



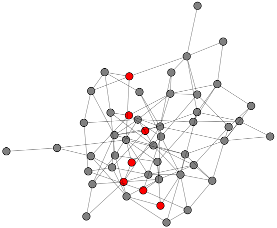
A = adjacency matrix of
Erdös-Rényi graph $p_{\text{edge}} = 0.1$

$n = 50$ nodes, $T = 10$,
$m = 7$ (rand. chosen) control nodes

U_{ij} i.i.d. r.v.'s, $\mathbb{E}[U_{ij}] = 0$, x_{f} rand. chosen



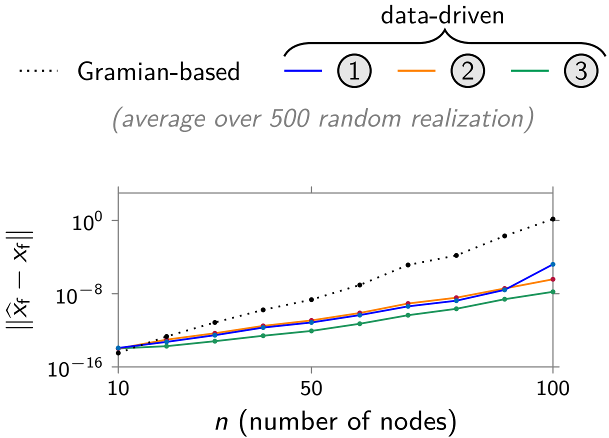
A numerical example



A = adjacency matrix of
Erdös-Rényi graph $p_{\text{edge}} = \frac{\ln n}{n} + 0.05$

$T = 2n$, $N = mT + 20$ data samples
$m = 7$ (rand. chosen) control nodes

U_{ij} i.i.d. r.v.'s, $\mathbb{E}[U_{ij}] = 0$, x_f rand. chosen



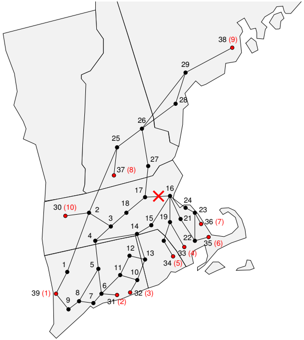
On some relevant extensions

- Data-driven formulas of minimum-energy controls can be established for data comprising experiments of different time lengths and/or initial conditions

- If data is corrupted by i.i.d. noise with known second-order statistics, asymptotically correct data-driven expressions of optimal control inputs can be derived

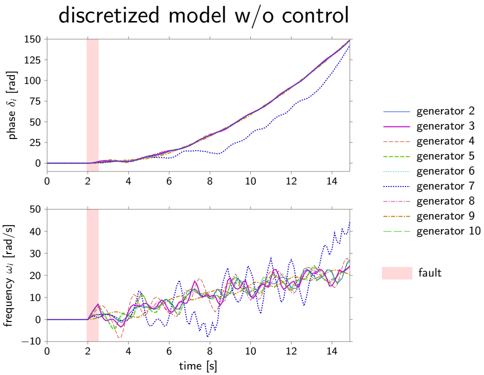
- The data-driven framework can be extended to control an output $y(t) \neq x(t)$ and to other cost functions depending on the input/state/output

A non-linear application

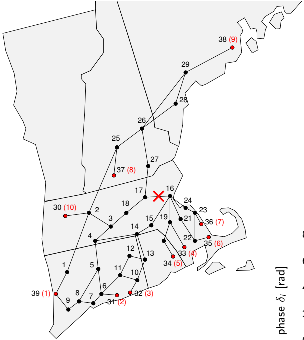


$$\dot{\delta}_i = \omega_i,$$

$$\frac{H_i}{\pi f_b}\dot{\omega}_i = -D_i\omega_i + P_{mi} - G_{ii}E_i^2 + \sum_{j=1, j\neq i}^{10} E_iE_j\left(G_{ij}\cos(\delta_i - \delta_j) + B_{ij}\sin(\delta_i - \delta_j)\right)$$



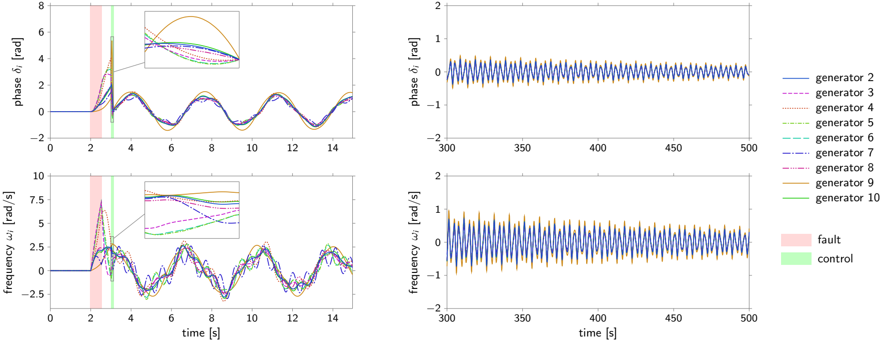
A non-linear application



$$\dot{\delta}_i = \omega_i,$$

$$\frac{H_i}{\pi f_b}\dot{\omega}_i = -D_i\omega_i + P_{mi} - G_{ii}E_i^2 + \sum_{j=1, j\neq i}^{10} E_iE_j\left(G_{ij}\cos(\delta_i - \delta_j) + B_{ij}\sin(\delta_i - \delta_j)\right) + u(t)$$

discretized model w/ control



Roadmap

Network controllability:
- the structural approach
- the “practical” approach

Non-normal dynamics and network structure

Data-driven control of networks

Conclusions & open challenges

Key takeaways

- Structural controllability ignores the role of edge weights and does not capture the “physical” degree of controllability of a network.

- In practice, to evaluate the controllability of a network, one should look at the energy required to control it (and so at the controllability Gramian).

- When using a limited number of control nodes, normal networks are difficult to control. By contrast, there are non-normal networks that are easy to control.

- When controlling a network, exact knowledge of network structure is not always necessary. One can design controls directly from experimental data.

Some interesting open problems

- "Interesting" classes of easy-to-control networks?
  (Relation to solution of Lyapunov equations, spectrum of Cauchy-like matrices,...)
- Control energy bounds for continuous-time networks?
  (In continuous-time, control energy always grows, at least linearly, with n !)
- Finite sample performance of noisy data-driven controls?
  (Tools from non-asymptotic random matrix theory?)
- Data-driven control of non-linear networks?
  (Map data to higher-dimensional, linear space? Koopman operator framework?)

Thank you !

Joint work with: S. Zampieri (UniPD), F. Pasqualetti (UCR), D. Bassett (UPenn)

Pasqualetti, Zampieri, Bullo, "Controllability metrics, limitations and algorithms for complex networks", IEEE TCNS, 2014

Baggio, Zampieri, "On the relation between non-normality and diameter in linear dynamical networks", ECC, 2018

Baggio, Bassett, Pasqualetti, "Data-driven control of complex networks", arXiv, 2020